General scaling method for electromagnetic fields with application to a matching problem

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A scaling method that reduces an electromagnetic problem described by a complicated geometry in a complicated medium to one described by a simple Cartesian geometry in a simple medium is explored and developed. This method creates and identifies an equivalent class of problems and their solutions from the Cartesian simple medium problem. We illustrate the usefulness of the method by applying it to the design of a reflectionless, distortionless, loaded matching section connecting a cylindrical and a conical coaxial waveguide, with the TEM fields being explicitly found everywhere.

I. INTRODUCTION AND SUMMARY

The transform methods in mechanics and fluid dynamics, which can carry a problem and its solution into a class of equivalent problems and solutions has been of interest for many years.[1] However, the application in electromagnetic (EM) theory of a scaling method of such similar nature has not received extensive attention and only a few works have recently been devoted to it.[2] The purpose of the present work is to investigate and develop for EM theory such a similarity or scaling transform.

In Sec. II we first develop the general theory of scaling, carrying an EM problem P into an equivalent P′, with the accompanying transformations for media, geometry, and fields. The advantage of such a procedure is, hopefully, to make the complexities of the geometry and the medium "cancel" each other in such a way that the resulting fields are simple and have known solutions. Also, some special cases of interest for 3-geometry[3] are discussed with results listed in Sec. II. Section III is devoted to the solving of a matching problem between a coaxial cylindrical and a conical waveguide.

II. THEORY OF ELECTROMAGNETIC SCALING

A. General theory

Basically, scaling is possible because the covariant divergences of the antisymmetric EM field tensors can be rewritten as ordinary divergences by including the metric determinant.[4] Let us consider a coordinate system $\{x^0 \equiv t, x^1, x^2, x^3\}$ with invariant length ds^2 and metric coefficients $g_{\mu\nu}$ (see Ref. 5):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \qquad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

where the summation convention, as in the whole text, is employed for repeated indices unless otherwise stated. For observers $\{0\}$ fixed in this frame $\{x^\mu\}$ with their spatial locations $x^i \equiv$ fixed, the Maxwell equations are[6]

$$\left[\sqrt{-g}\,e^j_{(i)}\frac{B^{(i)}}{\sqrt{g_{00}}}\right]_{,j} + [\sqrt{-g}\,\eta^{ijk}e^0_{(i)}e^l_{(j)}E_{(k)}]_{,l} = 0 \tag{2}$$

$$[\sqrt{-g}\,\eta^{ikl}e^j_{(i)}e^\nu_{(k)}E_{(l)}]_{,\nu} = \left[\sqrt{-g}\,e^j_{(i)}\frac{B^{(i)}}{\sqrt{g_{00}}}\right]_{,0}, \tag{3}$$

$$\left[\sqrt{-g}\,e^j_{(l)}\frac{D^{(l)}}{\sqrt{g_{00}}}\right]_{,j} - [\sqrt{-g}\,\eta^{ilk}e^0_{(i)}e^j_{(l)}H_{(k)}]_{,j} = \sqrt{-g}\left[\frac{q}{\sqrt{g_{00}}} + J^{(i)}e^0_{(i)}\right] \tag{4}$$

$$-[\sqrt{-g}\,\eta^{ikl}e^j_{(i)}e^\nu_{(k)}H_{(l)}]_{,\nu} = \left[\sqrt{-g}\,e^j_{(i)}\frac{D^{(i)}}{\sqrt{g_{00}}}\right]_{,0} + \sqrt{-g}\,J^{(i)}e^j_{(i)}. \tag{5}$$

Here we have used the following notation: $g \equiv \det(g_{\mu\nu})$; $\eta^{ijk} \equiv 0$ if (ijk) are not all different, ± 1 if (ijk) is an even or an odd order of (123); the Latin indices i, j, a, b, etc. equal 1, 2, 3; $e^\nu_{(\mu)}$ are the ν-contravariant components of the local comoving unit-tetrad vectors[7] $\mathbf{e}_{(\mu)}$ of $\{0\}$; $(\mathbf{D}, \mathbf{H})$, $(\mathbf{E}, \mathbf{B})$ are the usual EM field as seen by $\{0\}$; $D^{(i)}$, etc. are the physical vector components on $\mathbf{e}_{(i)}$ for $\{0\}$; $(q, \mathbf{J})$ are the physical charge-current source as seen by $\{0\}$; and $(\ )_{,\lambda} \equiv \partial/\partial x^\lambda(\ )$. Also we assume the medium to be linear with constitutive relations

$$D^{(i)} = \epsilon^{(i)}_{(j)}E^{(j)} + \alpha^{(i)}_{(j)}B^{(j)}, \tag{6}$$

$$H^{(i)} = \beta^{(i)}_{(j)}E^{(j)} + K^{(i)}_{(j)}B^{(j)}, \tag{7}$$

$$J^{(i)} = \sigma^{(i)}_{(j)}E^{(j)}. \tag{8}$$

If there are conducting boundaries, they are described by $F(\mathbf{x}) = 0$. The above, with appropriate boundary conditions, defines an EM problem P.

Now, a scaling can transform P into an EM problem P′ with simple Cartesian geometry and correspondingly scaled medium properties, sources, and boundary conditions as the following. If we define the scaled "fictitious" EM fields $(\mathbf{e}, \mathbf{b})$, $(\mathbf{h}, \mathbf{d})$ and their Cartesian components e^i, b^i, h^i, and d^i by

$$e^j \equiv \tfrac{1}{2}\sqrt{-g}\,\eta^{jab}\eta^{ikl}e^{(i)a}e^{(k)b}E^{(l)} \tag{9}$$

$$b^j \equiv \sqrt{-g}\,e^{(l)j}\left[\frac{B^{(l)}}{\sqrt{g_{00}}} + \eta^{ilk}e^{(i)0}E^{(k)}\right], \tag{10}$$

$$d^j \equiv -\sqrt{-g}\,e^{(l)j}\left[\frac{D^{(l)}}{\sqrt{g_{00}}} - \eta^{ilk}e^{(i)0}H^{(k)}\right] \tag{11}$$

$$h^j \equiv \tfrac{1}{2}\sqrt{-g}\,\eta^{jab}\eta^{ikl}e^{(i)a}e^{(k)b}H^{(l)} \tag{12}$$

in the coordinate frame $\{x^\mu\} \equiv \{t, x^1, x^2, x^3\}$, which is *now taken* to be Cartesian with

$$dS^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{13}$$

then the Maxwell equations (2) to (5) become respectively

$$\begin{cases} \nabla\cdot\mathbf{b} = 0 & (14) \\ \nabla\times\mathbf{e} = -\dfrac{\partial\mathbf{b}}{\partial t}, & (15) \end{cases}$$

$$\begin{cases} \nabla\cdot\mathbf{d} = \rho & (16) \\ \nabla\times\mathbf{h} = \mathbf{j} + \dfrac{\partial}{\partial t}\mathbf{d}, & (17) \end{cases}$$

as in the simple Cartesian sense. The scaled medium, corresponding to (6) to (8), is still linear and has as constitutive relations

$$\begin{aligned} d^j &= \xi^{js}e^s + A^{jb}b^b \\ &\equiv -e^{(l)j}\hat{e}^{(n)b}\Big\{\hat{e}^{(m)a}\eta^{abs}\Big[\frac{1}{2}\eta^{kmn}\Big(\frac{\epsilon^{(l)}_{(k)}}{\sqrt{g_{00}}} - \eta^{ilp}e^{(i)0}\beta^{(p)}_{(k)}\Big) \\ &\quad - \sqrt{g_{00}}e^{(m)0}\Big(\frac{\alpha^{(l)}_{(m)}}{\sqrt{g_{00}}} - \eta^{ilp}e^{(i)0}K^{(p)}_{(n)}\Big)\Big]e^s \\ &\quad - \sqrt{g_{00}}\Big[\frac{\alpha^{(l)}_{(n)}}{\sqrt{g_{00}}} - \eta^{ilp}e^{(i)0}K^{(p)}_{(n)}\Big]b^b\Big\}, \end{aligned} \tag{18}$$

$$\begin{aligned} h^j &= B^{js}e^s + \lambda^{jt}b^t \\ &\equiv \frac{1}{2}\eta^{jab}\eta^{ikl}e^{(i)a}e^{(k)b}\hat{e}^{(p)t}\Big[\hat{e}^{(m)r}\eta^{rts}\Big(\frac{\beta^{(l)}_{(n)}}{2}\eta^{nmp} \\ &\quad - \sqrt{g_{00}}K^{(l)}_{(p)}e^{(m)0}\Big)e^s - \sqrt{g_{00}}K^{(l)}_{(p)}b^t\Big], \end{aligned} \tag{19}$$

$$j^k = \Sigma^{kt}e^t \equiv -\tfrac{1}{2}e^{(i)k}\sigma^{(i)}_{(j)}\eta^{jmn}\hat{e}^{(m)a}\hat{e}^{(n)b}\eta^{abt}e^t, \tag{20}$$

where $e^{(i)k} \equiv -e^k_{(i)}$ and $\hat{e}^{(m)a} \equiv [e^{(i)j}]^{-1}$ exist since $\det[e^{(i)j}] \neq 0$. The source is scaled by[8]

$$\begin{cases} \rho = \sqrt{-g}\Big(\dfrac{q}{\sqrt{g_{00}}} + J^{(i)}e^0_{(i)}\Big) \\ j^k = \sqrt{-g}J^{(i)}e^k_{(i)}. \end{cases} \tag{21}$$

The corresponding boundary conditions are given through (9)–(12) and (18)–(19) to regulate the boundary behavior of the "fictitious" fields at the *same mathematical* boundaries with $\{x^\mu\}$ *interpreted as Cartesian coordinates*. For example, conducting boundaries are still described by the surfaces $F(\mathbf{x}) = 0$, on which $\mathbf{e}$ satisfies $N^i\hat{e}^{(i)j}\eta^{kjm}e^m\hat{e}^{(l)k} = 0$ where N^i is normal to $F(\mathbf{x}) = 0$.

Apparently, the reduction of P to P′ with greatly simplified geometry and differential equations is achieved at the expense of the much complicated medium properties. However, we must realize first that the "fictitious" fields and the "fictitious" problem P′ are the equivalent of and as real as the "real" problem P, and they can play a reverse role at our disposal.[9] Thus we can require the apparently complicated medium properties (18)–(20) to be simple enough so that we know the solution of the EM problem P′. Then, through the inverse scaling P′ → P, we obtain the whole class of P and their solutions with each of them corresponding to a particular choice of $g_{\mu\nu}$.

The inverse scaling P′ → P has, corresponding to (9)–(12),

$$\begin{cases} E^{(l)} = (1/2\sqrt{-g})\eta^{lmn}\hat{e}^{(m)a}\hat{e}^{(n)b}\eta^{abs}e^s & (22) \\ B^{(l)} = (-\sqrt{g_{00}}/\sqrt{-g})\{\hat{e}^{(l)j}b^j + e^{(m)0}\hat{e}^{(m)a}\hat{e}^{(l)b}\eta^{abs}e^s\}, & (23) \end{cases}$$

$$\begin{cases} D^{(l)} = (-\sqrt{g_{00}}/\sqrt{-g})\{\hat{e}^{(l)j}d^j - e^{(m)0}\hat{e}^{(m)a}\hat{e}^{(l)b}\eta^{abs}h^s\} & (24) \\ H^{(l)} = (1/2\sqrt{-g})\eta^{lmn}\hat{e}^{(m)a}\hat{e}^{(n)b}\eta^{abs}h^s, & (25) \end{cases}$$

and corresponding to (18) to (20)

$$\begin{aligned} D^{(l)} = &-\sqrt{g_{00}}\hat{e}^{(l)b}e^{(p)k}\{[\tfrac{1}{2}(\xi^{bi} \\ &- e^{(m)0}\hat{e}^{(m)a}\eta^{abc}B^{ci})\eta^{ikd}\eta^{pns}e^{(n)d} \\ &- (A^{bk} - e^{(m)0}\hat{e}^{(m)a}\eta^{abc}\lambda^{ck})\eta^{ips}e^{(i)0}]E^{(s)} \\ &- (1/\sqrt{g_{00}})(A^{bk} - e^{(m)0}\hat{e}^{(m)a}\eta^{abc}\lambda^{ck})B^{(p)}\}, \end{aligned} \tag{26}$$

$$\begin{aligned} H^{(l)} = \tfrac{1}{2}\eta^{lmn}\hat{e}^{(m)a}\hat{e}^{(n)b}\eta^{abc}e^{(i)s}\{\eta^{inp}[&\frac{B^{cj}}{2}\eta^{jsd}e^{(n)d} \\ &+ \lambda^{cs}e^{(n)0}]E^{(p)} - \lambda^{cs}[B^{(i)}/\sqrt{g_{00}}]\}, \end{aligned} \tag{27}$$

$$J^{(l)} = -\tfrac{1}{2}\hat{e}^{(l)k}\Sigma^{kj}\eta^{jab}\eta^{imn}e^{(i)a}e^{(m)b}E^{(n)}. \tag{28}$$

Also, the inverse of (21) is

$$\begin{cases} q = (\sqrt{g_{00}}/\sqrt{-g})(\rho - \hat{e}^{(i)k}e^{(i)0}j^k) \\ J^{(i)} = (-1/\sqrt{-g})\hat{e}^{(i)k}j^k. \end{cases} \tag{29}$$

B. Special geometries and media

For the P → P′ scaling, the mixed constitutive terms A^{ij} and B^{ij} in (18) and (19) are caused partly by the medium's intrinsic constitutive mixture $\alpha^{(i)}_{(j)}$ and $\beta^{(i)}_{(j)}$ in (6) and (7), and partly by the nontime-orthogonality term $e^{(i)0}$ of the frame $\{x^\mu\}$ with $g_{0i} \neq 0$. If both $\alpha^{(i)}_{(j)} = \beta^{(i)}_{(j)} = 0$ for the $\{x^\mu\}$ of problem P, then P′ has the simplified version of (18) and (19),

$$d^j = -e^{(l)}\hat{e}^{(m)a}\hat{e}^{(n)b}\eta^{abs}(\eta^{kmn}/2\sqrt{g_{00}})\epsilon^{(l)}_{(k)}e^s, \tag{30}$$

$$h^j = (-\sqrt{g_{00}}/2)\eta^{jab}\eta^{ikl}e^{(i)a}e^{(k)b}\hat{e}^{(p)t}K^{(l)}_{(p)}b^t. \tag{31}$$

If, in addition to the above, the original frame $\{x^\mu\}$ has a diagonal metric, i.e., $g_{\mu\nu} \equiv 0$ for $\mu \neq \nu$, we have[10]

$$e^\nu_{(\mu)} = \delta^\nu_\mu/\sqrt{|g_{\mu\mu}|} \equiv \delta^\nu_\mu/g_\mu. \tag{32}$$

Then (30), (31), (20) of the scaled medium further reduce to

$$\begin{pmatrix} d^1 \\ d^2 \\ d^3 \end{pmatrix} = \frac{1}{\sqrt{g_{00}}}\begin{pmatrix} g_2g_3 & & \bigcirc \\ & g_3g_1 & \\ \bigcirc & & g_1g_2 \end{pmatrix}\begin{pmatrix} \epsilon^{(1)}_{(1)} & \epsilon^{(1)}_{(2)} & \epsilon^{(1)}_{(3)} \\ \epsilon^{(2)}_{(1)} & \epsilon^{(2)}_{(2)} & \epsilon^{(2)}_{(3)} \\ \epsilon^{(3)}_{(1)} & \epsilon^{(3)}_{(2)} & \epsilon^{(3)}_{(3)} \end{pmatrix} \times \begin{pmatrix} 1/g_1 & & \bigcirc \\ & 1/g_2 & \\ \bigcirc & & 1/g_3 \end{pmatrix}\begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix}, \tag{33}$$

$$b^i \equiv \eta^{ij}h^j = [\text{in (33), replace } e^i \text{ by } h^i \text{ and } \epsilon^{(i)}_{(j)} \text{ by } \mu^{(i)}_{(j)}], \tag{34}$$

$$(1/\sqrt{g_{00}})j^i = [\text{in (33), replace } \epsilon^{(i)}_{(j)} \text{ by } \sigma^{(i)}_{(j)}], \tag{35}$$

where $\mu \equiv \kappa^{-1}$ such that $B^{(i)} = \mu^{(i)}_{(j)} H^{(j)}$. For this case, (9)–(12) which link fields of P′ and P reduce simply to (no summation here)

$$e^j = g_0 g_j E^{(j)} \tag{36}$$
$$b^j = g_1 g_2 g_3 [B^{(j)}/g_j], \tag{37}$$
$$d^j = (g_1 g_2 g_3/g_j) D^{(j)} \tag{38}$$
$$h^j = g_0 g_j H^{(j)}. \tag{39}$$

The inverse of (33)–(39) for P′→ P is obvious.

C. Remarks on medium restricted scalings in Euclidean 3-space

Physically limiting ourselves to certain class of media puts a restriction to the scaling. Preparing for a particular application we will treat in Sec. III, let us examine such limits in the following in a Euclidean 3-space and choosing *orthogonal coordinates* with $g_0 \equiv 1$.

If we require both P and P′ to have isotropic media, i.e., $\epsilon^{(i)}_{(j)} = -\epsilon\delta^{ij}$ and $\xi^{ij} = -\xi\delta^{ij}$, etc., then (33)–(35) imply

$$g_1 = g_2 = g_3 \tag{40}$$

and

$$\xi/\epsilon = \eta/\mu = \Sigma/\sigma = g_1. \tag{41}$$

In Euclidean 3-space, there exist only two[11] such coordinate frames, namely: Cartesian with $dS^2 = dt^2 - dx^2 - dy^2 - dz^2$, and inverse sphere with $dS^2 = dt^2 - a^4(dx'^2 + dy'^2 + dz'^2)/(x'^2 + y'^2 + z'^2)^2$.

If we require both P and P′ to have uniaxial media, i.e.,

$$\epsilon^{(i)}_{(j)} = \begin{pmatrix} \epsilon & & \\ & \epsilon & \\ & & \epsilon_3 \end{pmatrix}, \quad \xi^{ij} = \begin{pmatrix} \xi & & \\ & \xi & \\ & & \xi_3 \end{pmatrix}, \text{ etc.}, \tag{42}$$

then (33)–(35) imply

$$g_1 = g_2, \tag{43}$$
$$\xi/\epsilon = \eta/\mu = \Sigma/\sigma = g_3, \tag{44}$$
$$\xi_3/\epsilon_3 = \eta_3/\mu_3 = \Sigma_3/\sigma_3 = g_1^2/g_3. \tag{45}$$

For this case, the wave "impedances" $(\mu/\epsilon)^{1/2}$ and $(\mu_3/\epsilon_3)^{1/2}$ are unchanged in the scaling. This property will be used in Sec. III. Also, a simple calculation gives the results that we can obtain from any orthogonal coordinates (v^1, v^2, v^3) with $d\sigma^2 = f_1^2(dv^1)^2 + f_2^2(dv^2)^2 + f_3^2(dv^3)^2$ a coordinate system (x^1, x^2, x^3) having

$$d\sigma^2 = f_2^2[(dx^1)^2 + (dx^2)^2] + \{f_3^2(dx^3)^2/[F'(x^3)]^2\} \tag{46}$$
$$x^1 \equiv \int (f_1/f_2)dv^1 + \text{const}, \qquad x^2 \equiv v^2, \qquad x^3 \equiv F(v^3), \tag{47}$$

if[12]

$$f_1/f_2 = \text{function of } v^1 \text{ only}. \tag{48}$$

If we require only that both P and P′ have diagonal media, then no restriction on g_i is imposed. As a trivial example of this and to illustrate the P′→ P procedure, consider P′ as a parallel plate waveguide with plate at $x = a$ and $x = b$, filled with uniform η, ξ-simple medium, and having a TEM z-propagating wave $E^{(x)} = (\eta/\xi)^{1/2} H^{(y)} = \exp(i\omega\sqrt{\xi\eta}\, z)$. Take the scaling $(x, y, z) \to (\theta, \phi, r)$ as spherical coordinate, then (33)–(39) right away gives a legitimate P and its solution. The P is a conical waveguide with cones at $\theta = a/c_1$ and $\theta = b/c_1$ and filled with diagonal simple medium

$$\frac{\epsilon^{(i)}_{(j)}}{\xi} = \frac{\mu^{(i)}_{(j)}}{\eta} = \begin{pmatrix} c_2/(c_1 \sin\theta) & & \\ & (c_1 \sin\theta)/c_2 & \\ & & c_1 c_2/(r^2 \sin\theta) \end{pmatrix} \tag{49}$$

and has a TEM propagation with $rE^{(\theta)}/c_1 = (\eta/\xi)^{1/2} (r \sin\theta/c_2) H^{(\phi)} = \exp(i\omega\sqrt{\xi\eta}\, r)$. Here c_1 and c_2 are arbitrary length constants.

III. PERFECT μ,ϵ-LOADED MATCH BETWEEN CYLINDRICAL AND CONICAL COAXIAL WAVEGUIDES

To illustrate the use of the scaling method, we here apply it to examine and obtain a reflectionless and distortionless perfect match between a cylindrical (region I) and a conical (region III) coaxial waveguide. The given situation P in region I (see Fig. 1), which has perfectly conducting boundaries at $\rho = A$ and $\rho = B$[13] and is filled with ϵ, μ-simple uniform medium, is a TEM incident wave with $E^{(\rho)} = H^{(\phi)}(\mu/\epsilon)^{1/2} = [\exp(i\omega\sqrt{\epsilon\mu}\, z - i\omega t)]/\rho$. The solution of the problem may not be unique, and it is the simplest solution we want to find. Of course, from here on we use Euclidean 3-space and choose $g_{00} \equiv 1$, $g_{0i} \equiv 0$.

A. Selection and design of the scaling match

For mathematical simplicity we examine only orthogonal coordinates, and because of the nature of the problem we choose rotational ones in 3-space. Also, since only TEM wave is of interest, we designate the x^3-direction as the propagation direction and can try the "transverse isotropy" (42)–(45) that leaves ϵ_3, μ_3 free as the simplest general possibility.

Now the problem is to find a common P′ that underlies

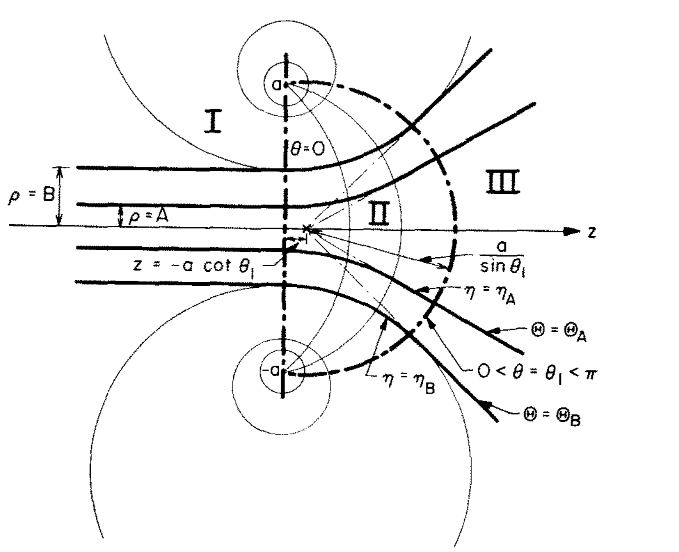

FIG. 1. Loaded perfect match between a cylindrical and a conical coaxial waveguide.

the whole range of regions I, III, and II which is the matching section to be found. Then a $P' \to P$ scaling will, hopefully, give back the desired result by choosing different scalings for different regions. In region I the P' as implied by the given P in region I is fixed. Through the use of (36)–(39) and (42)–(48) the P' is therefore specified by a TEM wave

$$e^1 = \exp(i\omega\sqrt{\mu\epsilon}\, x^3)/c_1, \tag{50}$$

$$h^2 = (\epsilon/\mu)^{1/2} \exp(i\omega\sqrt{\mu\epsilon}\, x^3)/c_1 \tag{51}$$

in a Cartesian coordinate (x^1, x^2, x^3) and in a Cartesian parallel plate waveguide which has boundaries at $x^1 = c_1 \ln(A/\rho_0)$ and $x^1 = c_1 \ln(B/\rho_0)$ and is filled with the medium

$$\frac{\xi^{ij}}{\epsilon} = \frac{\eta^{ij}}{\mu} = \frac{\Sigma^{ij}}{\sigma} = \begin{pmatrix} 1 & & \bigcirc \\ & 1 & \\ \bigcirc & & (\rho_0^2/c_1^2)\exp(2x^1/c_1) \end{pmatrix}. \tag{52}$$

Here in I the $(x^1, x^2, x^3) \equiv (c_1 \ln(\rho/\rho_0), c_1\phi, z)$ is obtained by using the cylindrical (ρ, ϕ, z) as the (v^1, v^2, v^3) to furnish the scaling, where c_1, ρ_0 are some length constants to be restricted later.

To choose a $P' \to P$ for region II from the above P', we first realize from the geometry of I and III that we need a coordinate system for (v^1, v^2, v^3) whose constant coordinate-surfaces can carry a plane into spheres convex w.r.t. the plane. The toroidal coordinates (η, ϕ, θ) provide just that.[14] As to region III, of course, we use the spherical coordinates (Θ, ϕ, r) as the (v^1, v^2, v^3).

Now, by using such (v^i) to convert the (x^i) of P' and by making sure that the x^i are continuous at junctions, simple straightforward calculations from (42) to (48) and the inverse of (36) to (39) give the required perfect matching. This match is shown in Fig. 1 and has the properties listed in Table I.

Notice that in the above table $F(\theta)$, $G(r)$ in general are dimensionless arbitrary smooth functions that satisfy $F(0) = 0$, $G(a/\sin\theta_1) = \tan\theta_1/2$, and the choices as shown are the results of requiring $\epsilon^{(1)}_{(1)} \geq \epsilon$ in II and $\epsilon^{(1)}_{(1)} = \epsilon$ in III. Also ρ_0 and a ($> B > A > 0$) are arbitrary length constants. Arrows at the top of the table denote boundaries that divide the regions.

B. Remarks and discussion

Firstly, since only TEM wave exists, the match can be realized by employing an isotropic medium using its transverse isotropy. Physically, it is obvious that there should be no reflection caused by μ, ϵ discontinuities and changes since η/ξ and therefore μ/ϵ is an invariant constant throughout the scaling. Also obviously, there should be no distortion because the smaller $(\mu\epsilon)$ near the inner matching conductor $\eta = \eta_A$ bends the plane phase front from the cylindrical I into a spherical phase front to match the spherical III.

TABLE I.

Quantities \ Regions	I	$z = 0 \to \leftarrow \theta = 0$ II	$\theta = \theta_1 \to \leftarrow r = a/\sin\theta_1$, $\theta < \theta_1 < \pi$ III
x^1	$a \ln(\rho/\rho_0)$	$a \ln[th(\eta/2)] + b$	$a \ln\left(\frac{\tan(\Theta/2)}{\tan(\theta_1/2)}\right) + b$
x^2	$a\phi$	$a\phi$	$a\phi$
x^3	z	$[\equiv aF(\theta)] = a \tan(\theta/2)$ and $F(0) = 0$	$[\equiv aG(r)] = r - a \cot\theta_1$ and $G(a/\sin\theta_1) = \tan(\theta_1/2)$
$g_1 = g_2$	(ρ/a)	$\sinh\eta/(\cosh\eta + \cos\theta)$	$r(\sin\Theta)/a$
g_3	1	$(\equiv 1/(\cosh\eta + \cos\theta)F'(\theta))$ $= (1 + \cos\theta)/(\cosh\eta + \cos\theta)$	$[\equiv 1/aG'(r)]$ 1
Boundaries			
$x^1 = a \ln(A/\rho)$ to $x^1 = a \ln(B/\rho_0)$	$\rho = A$ to $\rho = B$ $0 < A < B < a$	$\eta = 2 \tanh^{-1}A/a$ to $\eta = 2 \tanh^{-1}B/a$	$\Theta = 2 \tan^{-1}[(A/a) \tan(\theta_1/2)]$ to $\Theta = 2 \tan^{-1}[(B/a) \tan(\theta_1/2)]$
Media			
$\epsilon^{(i)}_{(j)}/\epsilon = \mu^{(i)}_{(j)}/\mu = \sigma^{(i)}_{(j)}/\sigma$ $\left[\epsilon^{(1)}_{(1)} = \epsilon^{(2)}_{(2)} = \xi/g_3\right.$ and $\left.\epsilon^{(3)}_{(3)} = \xi_3(g_3/g_1^2)\right]$	δ^{ij}	$\begin{pmatrix} \frac{(\cosh\eta + \cos\theta)}{(1 + \cos\theta)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \bigcirc \\ \bigcirc & \frac{\sinh\eta(1 + \cos\theta)}{(1 + \cosh\eta)^2} \end{pmatrix}$	δ^{ij}
	[now $\sigma^{(i)}_{(j)} \equiv 0$ for all regions, since $\sigma = 0$]		
Fields			
$\mathbf{E}\, e^{i\omega t}$	$\mathbf{e}_{(\rho)}\, (e^{i\omega\sqrt{\mu\epsilon} z}/\rho)$	$\mathbf{e}_{(\eta)} \frac{(\cosh\eta + \cos\theta)}{a \sinh\eta} \times e^{i\omega\sqrt{\mu\epsilon}\, a \tan(\theta/2)}$	$\mathbf{e}_{(\Theta)}(1/r \sin\theta) \times e^{i\omega\sqrt{\mu\epsilon}(r - a \cot\theta_1)}$
$\mathbf{H}\, e^{i\omega t}$	$\mathbf{e}_{(\phi)}\sqrt{\epsilon/\mu} \times \frac{e^{i\omega\sqrt{\mu\epsilon}\, z}}{\rho}$	$\mathbf{e}_{(\phi)} \sqrt{\frac{\epsilon}{\mu}}\, \frac{(\cosh\eta + \cos\theta)}{a \sinh\eta} \times e^{i\omega\sqrt{\mu\epsilon}\, a \tan(\theta/2)}$	$\mathbf{e}_{(\phi)}(\epsilon/\mu)^{1/2} \times \frac{e^{i\omega\sqrt{\mu\epsilon}(r - a \cot\theta_1)}}{r \sin\theta}$

The match so obtained is by no means unique. It is merely the simplest one. In fact, any rotational coordinates that can match smoothly with (ρ, ϕ, z) at left and (Θ, ϕ, r) at right can be used to provide (x^1, x^2, x^3) in II for $P' \to P$.

Concerning the realizability of the loading the required taper of μ is difficult. Since for normal incidence there is no reflection if and only if the impedance $(\epsilon/\mu)^{1/2}$ is constant and yet we need $(\mu\epsilon)$ to vary to furnish a transition, it is *impossible* to achieve a perfect match by an *orthogonal* scaling with only a varied ϵ and a fixed μ. But oblique incidence immediately suggests itself a Brewster angle tapering which may provide a perfect match through a nonorthogonal scaling with a fixed μ and varying ϵ. This and other possibilities of fixed-μ match are currently being investigated.

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[1]For general principle see, e.g., L. D. Landau and E. M. Lifshitz, *Fluid Dynamics* (Pergamon, New York, 1959), Secs. 19, 53, 118, 119; also, *Mechanics* (Pergamon, New York, 1960), Sec. 10; for an earlier example of such transforms, see E. J. Routh, Proc. Lond. Math. Soc. **12**, 73 (1881).

[2]For the theory of conformal mapped waveguides see, e.g., F. E. Borgnis and C. H. Papas, "Electromagnetic Waveguides and Resonators," in *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), 16, 358; for some examples see F. J. Tischer, Proc. IEEE 51, 1050 (1963); 53, 168 (1965); also J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 217; also P. Krasnooshkin, J. Phys. USSR **10**, 434 (1946); for an approach using invariance groups in differential forms, see B. K. Harrison and F. B. Estabrook, J. Math. Phys. **12**, 653 (1971); for a frequency scaling of reflection, see J. H. Davis and J. R. Cogdell, IEEE Trans. Antennas Propag. **19**, 58 (1971); for a scaling for reducing constantly moving uniform simple media, see R. J. Pogorzelski, IEEE Trans. Antennas Propag. **19**, 455 (1971).

[3]For most recent treatments on scaling see C. E. Baum, EMP Sensor and Simulation Notes DASA **32**, 1800 (1967), and Ph.D. thesis, Caltech Antenna Lab. Report **47**,California Institute of Technology (1968). For an example of tapering the dielectric to suit propagation, see P. L. Uslenghi, IEEE Trans. Antennas Propag. **17**, 644 (1969).

[4]See any text on relativistic electrodynamics, e.g., V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, New York, 1964), Sec. 24.

[5]Notice that the signature $(+---)$ is used therefore for special relativity flat space-time $dS^2 = dt^2 - dx^2 - dy^2 - dz^2$. Also geometrized MKS unit with $\mu_0 = \epsilon_0 = 1$ for vacuum is used for simple formalisms. For retrieval to MKS see, e.g., attached table in T. C. Mo, Radio Sci. **6**, 673 (1971). For any applications in special relativistic EM theory, only insert appropriate powers of c $(3 \times 10^8$ meters/sec) in the final answer to fix dimensions right.

[6]T. C. Mo, J. Math. Phys. **11**, 2589 (1970), Sec. 4. Also for 3-vectors, $D^{(l)} \equiv -D_{(l)}$, etc.

[7]See Ref. 6, Sec. 2; and any textbook with tensor calculus, e.g., J. L. Synge, *General Relativity* (Interscience, New York, 1960), Sec. 3.

[8]Notice that the $(\rho, \mathbf{j})/\sqrt{g_{00}}$ of this scaled current-density is the contravariant 4-vector component J^μ in the frame $\{x^\mu\}$ of (1).

[9]For both P and P′ to be really physical, care must be taken into account to make x^i have dimensions of length such that $g_{\mu\nu}$ are dimensionless pure numbers.

[10]Here we use g_μ in the diagonal $dS^2 = g_0^2 dt^2 - (g_i)^2 (dx^i)^2$, instead of the conventional notation $dS^2 = h_0^2 - (h_i)^2 (dx^i)^2$ just to avoid confusion with the scaled magnetic field $\mathbf{h}$.

[11]Ref. 3, Antenna Report **47**, p. 76.

[12]Notice that for geometry (46), $x^3 = \text{const}$ is either a sphere or a plane; see Ref. 11, p. 83; and L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces* (Dover, New York, 1960), p. 449.

[13]The ρ in cylindrical $(\rho\phi z)$ coordinates from here on should not cause confusion with the previous charge density ρ.

[14]P. Moon and D. E. Spencer, *Field Theory Handbook* (Springer-Verlag, Berlin, 1961), p. 112.